On the uniqueness of the Moyal structure of phase-space functions

1997 J. Phys. A: Math. Gen. 30 4857

(http://iopscience.iop.org/0305-4470/30/13/033)

J. Phys. A: Math. Gen. **30** (1997) 4857–4866. Printed in the UK PII: S0305-4470(97)80514-8

On the uniqueness of the Moyal structure of phase-space functions

C Tzanakis† and A Dimakis‡

† University of Crete, 74 100 Rethymnon, Crete, Greece

‡ University of the Aegean, Department of Mathematics, 83 200 Karlovasi, Samos, Greece

Received 17 December 1996

Abstract. Uniqueness up to isomorphism, of the Moyal product and bracket of functions on $\mathbb{R}^{2n}$ as associative and Lie deformations of the ordinary product and Poisson bracket, is known to follow under additional hypotheses. Using an integral formalism we show here that this result holds without these hypotheses.

1. Introduction

It is well known that quantum operators can be mapped to phase-space functions and vice versa, in many different ways, depending on the ordering rule chosen for the operator to which the monomial $q^n p^m$ is mapped (see e.g. [1]). Originally this was done by the Weyl transformation corresponding to a symmetric ordering [2]. The inverse of this mapping, the Wigner transformation, was originally devised in order to formulate quantum expectation values as classical averages on the phase space Γ of the system under consideration [3]. Since the work of Moyal [4], who showed the relation between these two mappings, many other ordering rules have been considered (see e.g. [5, 6]), corresponding to some generalization of the Wigner transformation, or equivalently, of the Weyl symmetric ordering, *depending on the physical problem under consideration* (see e.g. [5] for a discussion of the advantages of different choices of ordering in different problems). This is due to the fact that such transformations (see (1.1) and (1.2) below) make possible a phase-space formulation of quantum theory, which has been proved to be convenient in a wide variety of domains: from quantum optics [7], kinetic and transport theory [8], and scattering problems ([9, 5 section 11.1] and references therein), to string theory [10], and the study of chaotic and ergodic behaviour in quantum systems ([5] section 11.2 and references therein). A large class of such transformations is given by (e.g. [1, 11])§

$$\Omega_g\colon A(\boldsymbol{q},\boldsymbol{p}) \to \Omega_g(A) = \hat{A} := \frac{1}{(2\pi)^n}\int \mathrm{d}\sigma\, \Omega(\sigma)\tilde{A}(\sigma)\,\mathrm{e}^{\mathrm{i}\sigma\cdot\hat{z}} \tag{1.1}$$

where we use the following notation: phase-space coordinates $(\boldsymbol{q},\boldsymbol{p}) =: z$; corresponding quantum operators $(\hat{q},\, \hat{p} = -\mathrm{i}\hbar(\partial/\partial\boldsymbol{q})) =: \hat{z}$; quantum mechanical operators $\hat{A}$; phase-space functions $A(\boldsymbol{q},\boldsymbol{p})$; Fourier transform

$$\tilde{A}(\eta,\boldsymbol{\xi}) = \frac{1}{(2\pi)^n}\int \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\sigma\cdot z}A(z) \qquad \sigma = (\boldsymbol{\eta},\boldsymbol{\xi}).$$

§ It can be shown that any linear transformation of quantum operators to phase-space functions, which is phase-space translation invariant is the inverse of (1.1). The calculations will not be given here, but see [12], (15).

0305-4470/97/134857+10$19.50

Here and in what follows we consider a $2n$-dimensional flat phase-space, $\Gamma = \mathbb{R}^{2n}$ where $\sigma \cdot z$ denotes its scalar product, and $\Omega(\sigma)$ is assumed to be an entire analytic function of σ without zeros. Then Ω_g^{-1} exists formally and in Dirac notation it is given by

$$\Omega_g^{-1}\colon \hat{A} \rightarrow A(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{\pi^n} \int \mathrm{d}\boldsymbol{q}'\, \mathrm{d}\boldsymbol{p}'\, \mathrm{d}\boldsymbol{t}\, \mathrm{e}^{-\boldsymbol{p}'\cdot\boldsymbol{t}/\mu} \omega(\boldsymbol{q}-\boldsymbol{q}', \boldsymbol{p}-\boldsymbol{p}') \langle \boldsymbol{q}'-\boldsymbol{t}|\hat{A}|\boldsymbol{q}'+\boldsymbol{t}\rangle \tag{1.2}$$

where

$$\omega(z) := \frac{1}{(2\pi)^n} \int \mathrm{d}\sigma\, \frac{\mathrm{e}^{\mathrm{i}\sigma\cdot z}}{\Omega(\sigma)} \qquad \mu := \frac{\mathrm{i}\hbar}{2}. \tag{1.3}$$

For $\Omega = 1$, (1.1) and (1.2) give the Weyl and Wigner transformations respectively ([2, 3]) so that in the general case we may call them generalized Weyl and Wigner transformations (GWT) respectively. Sometimes, A in (1.2) is called a smoothed Wigner distribution, corresponding to $\hat{A}$ with smoothing kernel ω, since it has all the basic properties of a Wigner distribution under quite general conditions (by (1.2) A is the convolution of ω with the Wigner transform of $\hat{A}$). Therefore such distributions have also been considered in quantum statistical mechanics, especially in connection with the question of whether they can be interpreted as probability densities when $\hat{A}$ is a density matrix, specifically if they are non-negative (a condition violated in general by Wigner distributions) [13].

On the other hand, it is well known that a GWT induces on the vector space of C^∞ phase-space functions $F(\Gamma)$, the structure of an associative, in general non-Abelian algebra, and of a Lie algebra via

$$f \star_\Omega g := \Omega_g^{-1}(\Omega_g(f)\Omega_g(g)) \tag{1.4}$$

$$[f, g]_\Omega := \frac{1}{2\mu}(f \star_\omega g - g \star_\Omega f) \tag{1.5}$$

respectively, with $f, g \in F(\Gamma)$. For $\Omega = 1$ (1.4) and (1.5) give the Moyal product and bracket denoted by $\star$, [,], [4]. In the classical limit $\mu \rightarrow 0$, the latter reduces to the Poisson bracket.

Equations (1.4) and (1.5) can explicitly be written as

$$(f \star_\Omega g)(z) = \frac{1}{(2\pi)^{2n}} \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \tilde{f}(\sigma)\tilde{g}(\sigma') B(\sigma, \sigma')\, \mathrm{e}^{\mathrm{i}(\sigma+\sigma')\cdot z} \tag{1.6}$$

$$[f, g]_\Omega(z) = \frac{1}{(2\pi)^{2n}} \int \mathrm{d}\sigma\, d\sigma'\, \tilde{f}(\sigma)\tilde{g}(\sigma') A(\sigma, \sigma')\, \mathrm{e}^{\mathrm{i}(\sigma+\sigma')\cdot z} \tag{1.7}$$

where

$$B(\sigma, \sigma') = \frac{\Omega(\sigma)\Omega(\sigma')}{\Omega(\sigma+\sigma')}\, \mathrm{e}^{\mu\,\sigma'\wedge\sigma} \tag{1.8}$$

$$A(\sigma, \sigma') = \frac{\Omega(\sigma)\Omega(\sigma')}{\Omega(\sigma+\sigma')} \frac{\sinh(\mu\sigma'\wedge\sigma)}{\mu} \tag{1.9}$$

with $\sigma' \wedge \sigma := J_{ij}\sigma'^i\sigma^j$, where J_{ij} is the canonical symplectic matrix of $\mathbb{R}^{2n}$, that is

$$J = \begin{pmatrix} 0 & (\delta_{ij}) \\ -(\delta_{ij}) & 0 \end{pmatrix}.$$

These imply that the mapping U,

$$f \rightarrow Uf\colon (Uf)(z) = \int \mathrm{d}\sigma\, \Omega(\sigma)\tilde{f}(\sigma)\, \mathrm{e}^{\mathrm{i}\sigma\cdot z} \tag{1.10}$$

is an algebra and Lie-algebra isomomorphism of $(F(\Gamma), \star_\Omega, [,]_\Omega)$ and $(F(\Gamma), \star, [,])$†.

Therefore, although (1.6) and (1.7) define binary operations of functions in classical phase-space with different physical interpretations that are very useful in various problems of quantum physics (see the previous references), the underlying abstract algebraic structure is independent of Ω. Thus the question naturally arises, *whether more general binary operations are defined via (1.6) and (1.7), which are respectively an associative product and a Lie product and for which the corresponding kernels are not of the form (1.8) and (1.9)* and if so, to characterize the corresponding algebras.

It is the aim of this paper to study this problem in its general form. Special cases have been treated in the literature. To see this, *we introduce the notation* $\star_B$, $[,]_A$ for (1.6) and (1.7) *with* B, A *not given a priori* by (1.8) and (1.9), assuming that

(i) B, A are entire analytic functions of their arguments and B has no zeros, so that for some entire function $b(\sigma, \sigma')$ with $b_s(\sigma, \sigma')$ (respectively $b_a(\sigma, \sigma')$) symmetric (respectively antisymmetric) part

$$B(\sigma, \sigma') = \mathrm{e}^{b(\sigma,\sigma')} = \mathrm{e}^{b_s(\sigma,\sigma')}\,\mathrm{e}^{b_a(\sigma,\sigma')} \tag{1.11}$$

(ii) constants are in the centre of the Lie algebra, i.e.

$$[f, 1]_A = 0 \qquad \text{for all } f \in F(\Gamma). \tag{1.12}$$

Developing A in a power series, we formally get

$$[f, g]_A(z) = \sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\sum_{j=0}^{r}\sum_{k=0}^{s} b_{rj,sk}(\partial_q^j\partial_p^{r-j} f)(\partial_q^k\partial_p^{s-k} g) \tag{1.13}$$

where $r, s, j, k \in \mathbb{N}^n$ are differentiation indices, each sum abbreviates an n-fold summation and $b_{rj,sk}$ are given in terms of the derivatives of A at 0. A similar expression is obtained for $f \star_B g$.

For *b a polynomial*, a characterization of the $\star_B$-algebras has been given by Vey ([14] section 1), which shows that in this case *(1.8) is essentially unique*. In section 2 and appendix A *we will show* by elementary means, that *this result remains valid when b is an entire function*.

Vey also considered on arbitrary symplectic manifolds, deformations of the Poisson–Lie algebra of C^∞-functions for which

$$b_{rj,sk} = \begin{cases} 0 & \text{for } |r| \neq |s| \text{ or } |r| = |s| = 2m \\ \lambda^m b_{2m+1,jk} & \text{for } |r| = |s| = 2m+1 \qquad \lambda \in \mathbb{C} \end{cases} \tag{1.14}$$

with $|r| = r_1 + \cdots + r_n$ etc so that the coefficient of λ^m in (1.13) is a bidifferential operator of odd order with principal symbol identical to that of the Poisson bracket defined by the symplectic structure (Vey deformations [14], section 4, [15], p 74, [16], section 7). He showed that such essential (i.e. non-isomorphic to the original Poisson–Lie algebra) deformations exist as long as the third de Rahm cohomology is trivial([14], p 446). This restriction was later removed ([17]). On the other hand Lichnerowicz, Flato, Sternheimer

† From this it follows that a necessary and sufficient condition for $[,]_\Omega$ to reduce to the Poisson bracket, given that $(\boldsymbol{q}, \boldsymbol{p}) \overset{\Omega_g}{\to} (\hat{q}, \hat{p})$, is that $\lim_{\mu\to 0}\Omega(\sigma) = \Omega_0 = 1$. Then

$$\lim_{\mu\to 0} A(\sigma, \sigma') = \frac{\Omega_0(\sigma)\Omega_0(\sigma')}{\Omega_0(\sigma+\sigma')}\sigma' \wedge \sigma$$

and (1.10) with Ω_0, replacing Ω, is a Lie algebra isomorphic with the Poisson Lie algebra. We may call such algebras *generalized Poisson Lie algebras*, and consider them as limits of generalized Moyal algebras. We agree to include them in (1.9) for $\mu = 0$, and for the sake of brevity we will use only the term Moyal Lie algebra.

and coworkers proved that *if the manifold admits a flat symplectic connection (e.g.* $\mathbb{R}^{2n}$*), then the only non-trivial such deformation which is a formal function of the corresponding Poisson bracket, is the Moyal bracket* defined via the symplectic structure and this connection ([18], theorem 1, [15], theorem 5, [16], section 6). *The restriction to the algebra of polynomials* with complex coefficients in a symplectic vector space, implies that a sufficient condition for the deformation to be a function of the Poisson bracket, is that *the deformed bracket is invariant under all affine contact transformations* ([19], theorem 1.3), hence the uniqueness of the Moyal bracket follows again ([19], theorem 2.11). A similar uniqueness result holds for associative deformations as well ([19], theorem 3.5). The uniqueness of the Moyal bracket as a deformation of the Poisson bracket was shown to hold without the above conditions, only in the case of $\mathbb{R}^2$, by substituting (1.13) to the Jacobi identity and solving the resulting recurrence relations for the bs ([20]; however, the proof was made explicit only under (1.14) and it does not seem to be easily generalized in $\mathbb{R}^{2n}$ since the calculations become too complicated).

To the best of our knowledge, further work in this field has been concerned mainly with the development of analytical aspects of the Moyal structure of appropriately defined spaces of functions or distributions and applications to quantum physics; for instance, the phase-space description of spin systems [21], the quantization of classical conditionally periodic systems [22] or, more generally, of systems with a non-Euclidean phase space, [23, 34].

However important, the above beautiful considerations leave unanswered the *general* question of the characterization of the $[\,,\,]_A$-algebras in $\mathbb{R}^{2n}$, since they *presuppose* on the one hand the existence of a symplectic structure on the manifold and on the other hand (1.14). It is perhaps interesting that *the existence of a symplectic form and the invariance of the* $[\,,\,]_A$-*bracket under the corresponding symplectic group follow from the Jacobi identity alone*, i.e. that the latter alone implies that A in (1.7) is given by (1.9), hence (up to isomorphism) uniqueness of the Moyal bracket follows as well. This is the subject of section 3.

More precisely, using (1.6) and (1.7) we get

$$\begin{aligned}
f \star_B (g \star_B h)(z) &= \frac{1}{(2\pi)^{3n}} \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \mathrm{d}\sigma''\, \mathrm{e}^{\mathrm{i}(\sigma+\sigma'+\sigma'')\cdot z} \\
&\quad \times B(\sigma'', \sigma')B(\sigma''+\sigma', \sigma)\tilde{f}(\sigma)\tilde{g}(\sigma')\tilde{h}(\sigma'') \\
[f, [g, h]_A]_A(z) &= \frac{1}{(2\pi)^{3n}} \int \mathrm{d}\sigma\, \mathrm{d}\sigma'\, \mathrm{d}\sigma''\, \mathrm{e}^{\mathrm{i}(\sigma+\sigma'+\sigma'')\cdot z} \\
&\quad \times A(\sigma, \sigma'+\sigma'')A(\sigma', \sigma'')\tilde{f}(\sigma)\tilde{g}(\sigma')\tilde{h}(\sigma'')
\end{aligned}$$

hence associativity and the Jacobi identity are respectively found to be equivalent to

$$B(\sigma, \sigma')B(\sigma+\sigma', \sigma'') = B(\sigma, \sigma'+\sigma'')B(\sigma', \sigma'') \tag{1.15}$$

$$A(\sigma, \sigma'+\sigma'')A(\sigma', \sigma'') + A(\sigma', \sigma''+\sigma)A(\sigma'', \sigma) + A(\sigma'', \sigma+\sigma')A(\sigma, \sigma') = 0 \tag{1.16}$$

$$A(\sigma, \sigma') = -A(\sigma', \sigma). \tag{1.17}$$

In the rest of the paper we show by quite simple (sometimes even elementary) methods that essentially the only solutions of these functional relations are respectively (1.8) and (1.9). Thus, up to isomorphism, the uniqueness of the Moyal product and bracket is proved in a more general setting than previously.

2. The characterization of the $\star_B$-algebras

In this section we will show that (1.15) essentially implies (1.8)—the converse is trivial. Starting from (1.11), we may remark that the special case when b is a *polynomial* without constant term, was considered long ago by Vey, who showed using cohomological arguments that b_a is necessarily bilinear and that it determines uniquely such a $\star_B$-algebra, up to isomorphism ([14], section 1, proposition 2 and its corrolary).

In the following, using similar arguments, *we show that the restriction to a polynomial function for b is not necessary*. In fact, if we consider the extension E of the translation group $(\mathbb{R}^{2n}, +)$ by the multiplicative group $\mathbb{C}^* := \mathbb{C} - \{0\}$, i.e. a short exact sequence

$$1 \to \mathbb{C}^* \to E \to \mathbb{R}^{2n} \to 0$$

then by writing (σ, ζ) for an element of E and defining

$$(\sigma, \zeta)(\sigma', \zeta') := (\sigma + \sigma', B(\sigma, \sigma')\zeta\zeta') \tag{2.1}$$

we see that associativity of (2.1) is equivalent to (1.15) (cf [25], section 6.10, [14], (4), [10], (6)). Thus the determination of B, or for that matter b, is equivalent to the determination of all equivalence classes of extensions of $(\mathbb{R}^{2n}, +)$ by $\mathbb{C}^*$, which is in 1–1 correspondence with the second cohomology group $H^2(\mathbb{R}^{2n}, \mathbb{C}^*)$ (e.g. [25], theorem 6.15). In fact given (1.11), equation (1.15) becomes

$$b(\sigma', \sigma'') - b(\sigma + \sigma', \sigma'') + b(\sigma, \sigma' + \sigma'') - b(\sigma, \sigma') = 0. \tag{2.2}$$

It is not difficult to see that since $\mathbb{R}^{2n}$ acts trivially on $\mathbb{C}^*$, (2.2) says that b is a coboundary $(\delta b)(\sigma, \sigma', \sigma'') = 0$ of the complex $C(\mathbb{R}^{2n}, \mathbb{C}^*)$ of complex-valued functions without zeros, on $(\mathbb{R}^{2n})^k$, $k = 0, 1, 2, \ldots$. Thus b is determined up to $\delta\chi$, for some $\chi \in C^1(\mathbb{R}^{2n}, \mathbb{C}^*)$. Evidently

$$(\delta\chi)(\sigma, \sigma') = \chi(\sigma') - \chi(\sigma + \sigma') + \chi(\sigma). \tag{2.3}$$

In view of the above remarks, isomorphic classes of $\star_B$-algebras are in 1–1 correspondence with the elements of $H^2(\mathbb{R}^{2n}, \mathbb{C}^*)$. It is interesting to notice that *(2.2) can be solved by elementary means*. This is done in appendix A, where we show that $b_s = \delta\chi$ for some 1-cochain χ and b_a is a 2-form on $\mathbb{R}^{2n}$. Therefore, $\mathbb{R}^{2n}$ can be split as a sum of a space in which b_a is non-degenerate and its kernel. Consequently there exist (non-uniquely determined) coordinates such that $\sigma = (\rho, \tau)$ with $(\rho, 0) \in \operatorname{Ker} b_a$ and

$$B(\sigma, \sigma') = \frac{\Omega(\sigma)\Omega(\sigma')}{\Omega(\sigma + \sigma')} \mathrm{e}^{b_a(\sigma,\sigma')} \qquad \Omega(\sigma) = \mathrm{e}^{\chi(\sigma)} \qquad b_a(\sigma, \sigma') = \mu(\tau \wedge \tau') \tag{2.4}$$

for some $\mu \in \mathbb{C}$. Consequently, if the dual splitting of the phase-space coordinates is $z = (x, y)$, then it is easily seen that functions of x only—in other words functions which satisfy $b_a^{ij}(\partial_i f) = 0$—belong to the centre of $(F(\Gamma), \star_B)$. On the other hand, it is easy to see that by (2.4), non-degeneracy of b_a implies that the $\star_B$-algebra has a trivial center. Thus, the above results can be summarized in

Theorem 1. Any $\star_B$-associative algebra, having a trivial centre, and for which $B(\sigma, \sigma')$ is an entire analytic function without zeros, is a generalized Moyal algebra (1.6) and (1.8), hence by (1.10) it is isomorphic to the Moyal algebra defined by (1.6) and (1.8) with $\Omega = 1$.

As a final remark we notice that for b_a non-degenerate, the extension (2.1) of $\mathbb{R}^{2n}$ by $\mathbb{C}^*$ is the direct product of $\mathbb{R}^*$ with the Heisenberg group, a fact following from (2.1) in view of (2.1) ([26], section 15, particularly (15.2)).

3. The characterization of the $[\,,\,]_A$-Lie algebras

We next turn to the study of the Lie algebras defined by (1.7), i.e. to the study of (1.6), assuming that constants annihilate the Lie product, i.e. (1.12) holds, or equivalently

$$A(0,\sigma) = 0. \tag{3.1}$$

Differentiating (1.16) with respect to σ^i and putting $\sigma = 0$ we get that $\partial_i^1 A(0,\sigma)$ is linear, i.e.

$$\partial_i^1 A(0,\sigma) = \omega_{ij}\sigma^j \tag{3.2}$$

where ω_{ij} is antisymmetric and in this section we write $\partial_i^a A$ for the ith component of the gradient of A in the ath argument ($a = 1, 2$) and the summation convention is always assumed. Differentiating (1.16) with respect to σ^i and σ^j at $\sigma = 0$ gives

$$(X_{ij}(\sigma') + X_{ij}(\sigma''))A(\sigma',\sigma'') = (a_{ij}(\sigma') + a_{ij}(\sigma'') - a_{ij}(\sigma' + \sigma''))A(\sigma',\sigma'') \tag{3.3}$$

where

$$a_{ij}(\sigma) := \partial_i^1\partial_j^1 A(0,\sigma) \tag{3.3a}$$

$$X_{ij}(\sigma) := \sigma^k\omega_{k(i}\delta_{j)}^{\ell}\frac{\partial}{\partial\sigma^{\ell}}. \tag{3.3b}$$

To simplify the notation by supressing indices whenever it is necessary, we introduce symmetric parameters α^{ij}, β^{ij} and set

$$X_\alpha := \frac{1}{2}\alpha^{ij}X_{ij}(\sigma) = \sigma^k\omega_{ki}\alpha^{ij}\frac{\partial}{\partial\sigma^j} \tag{3.4a}$$

$$Z_\alpha(\sigma,\sigma') := X_\alpha(\sigma) + X_\alpha(\sigma') \tag{3.4b}$$

$$a_\alpha := \frac{1}{2}\alpha^{ij}a_{ij}(\sigma) \tag{3.4c}$$

$$\hat{a}_\alpha(\sigma,\sigma') := a_\alpha(\sigma) + a_\alpha(\sigma') - a_\alpha(\sigma+\sigma'). \tag{3.4d}$$

Thus (3.3) becomes

$$(Z_\alpha A)(\sigma',\sigma'') = (\hat{a}_\alpha A)(\sigma',\sigma''). \tag{3.5}$$

The crucial step is to notice that by (3.4*a*) and (3.4*b*)

$$[X_\alpha, X_\beta] = X_\gamma \tag{3.6a}$$

hence

$$[Z_\alpha, Z_\beta] = Z_\gamma \tag{3.6b}$$

with

$$\gamma^{ij} = \alpha^{ik}\omega_{k\ell}\beta^{\ell j} - \beta^{ik}\omega_{k\ell}\alpha^{\ell j}. \tag{3.7}$$

Therefore, provided ω_{ij} is non-degenerate, the $(2n^2+n)$-independent vector fields X_{ij} generate a Lie algebra, which is identical to the Lie algebra of the symplectic group of ω_{ij}, since

$$(X_{ij}(\sigma') + X_{ij}(\sigma''))\omega_{k\ell}\sigma'^k\sigma''^\ell = 0. \tag{3.8}$$

Since Z_α and Z_β are derivations, applying $[Z_\alpha, Z_\beta]$ to $A(\sigma',\sigma'')$ and using (3.6*b*) and (3.5), we get

$$Z_\alpha\hat{a}_\beta - Z_\beta\hat{a}_\alpha = \hat{a}_\gamma. \tag{3.9}$$

But, from (3.3b) and (3.4b) we have

$$Z_\alpha(\sigma', \sigma'')a_\beta(\sigma') = (X_\alpha\, a_\beta)(\sigma')$$
$$Z_\alpha(\sigma', \sigma'')a_\beta(\sigma' + \sigma'') = (X_\alpha a_\beta)(\sigma' + \sigma'')$$

hence (3.9) implies the linearity in σ of $X_\alpha a_\beta - X_\beta a_\alpha - a_\gamma := \sigma^i c_i(\alpha, \beta)$. Differentiating this with respect to σ^k at $\sigma = 0$ and putting

$$\tilde{a}_\alpha(\sigma) := a_\alpha(\sigma) - \sigma^i \partial_i a_\alpha(0) \tag{3.10}$$

we finally get

$$X_\alpha\, \tilde{a}_\beta - X_\beta\, \tilde{a}_\alpha = \tilde{a}_\gamma. \tag{3.11}$$

As a consequence of (3.11) and (3.6a), the system of first-order differential equations

$$X_{ij}\, \chi(\sigma) = \tilde{a}_{ij}(\sigma) \tag{3.12}$$

is locally integrable. Going back to (3.3) and using (3.12) we may rewrite it locally as

$$[X_{ij}(\sigma') + X_{ij}(\sigma'')] \left(\frac{\Omega(\sigma' + \sigma'')}{\Omega(\sigma')\Omega(\sigma'')} A(\sigma', \sigma'') \right) = 0 \tag{3.13}$$

where $\Omega(\sigma) := \mathrm{e}^{\chi(\sigma)}$. This means that $\Omega(\sigma'+\sigma'')A(\sigma', \sigma'')/\Omega(\sigma')\Omega(\sigma'')$ is invariant under the symplectic group of ω_{ij} (cf (3.8)), and therefore it is a function of $\omega(\sigma', \sigma'')$, since this is the only bilinear invariant of the group on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ (cf [27], appendix F). Therefore

$$A(\sigma, \sigma') = \frac{\Omega(\sigma)\Omega(\sigma')}{\Omega(\sigma + \sigma')} h(\omega(\sigma, \sigma')) \tag{3.14}$$

for some function h. Evidently h satisfies the Jacobi condition (1.16). Then by the lemma in appendix B, $h(x) = c \sinh \mu x$ or $h(x) = cx$ and therefore locally A has the form (1.9).

To get (3.14) *globally*, we notice that the symplectic group of ω_{ij} acts transitively on $\mathbb{R}^{2n} - \{0\}$ and consequently constants are the only invariants it has. Therefore if χ_a, χ_b are local solutions of (3.12) in some intersecting open subsets U_a, U_b of $\mathbb{R}^{2n}$, then by (3.12), in $U_a \cap U_b$

$$X_{ij}(\chi_a - \chi_b) = 0$$

hence $\chi_a = \chi_b + \kappa_{ab}$ for some constant κ_{ab}. Now since $\mathbb{R}^{2n}$ is simply connected we can write in a consistent way $\kappa_{ab} = \kappa_b - \kappa_a$ for all such pairs U_a, U_b of an open covering of $\mathbb{R}^{2n}$ and thus redefining χ_a to $\chi_a + \kappa_a = \chi_b + \kappa_b$ we obtain a global solution χ of (3.12), hence h in (3.14) is globally defined as well.

Evidently from our results so far, it follows that if $(F(\Gamma), [\ ,\ ]_A)$ has a non-trivial centre, then ω is degenerate. Conversely, suppose ω has a d-dimensional kernel then as in the previous section, we may write $\sigma = (\rho, \tau)$ with $(\rho, 0) \in \mathrm{Ker}\, \omega$, and similarly $z = (x, y)$. Take any function f of x, then its Fourier transform is $\tilde{f}(\rho)\delta(\tau)$ and a direct calculation shows that for any $(\rho_0, 0) \in \mathrm{Ker}\, \omega$

$$F(x) := \tilde{f}(0)(d - \mathrm{i}\rho_0 \cdot x) - (\partial_k \tilde{f})(0)\rho_0^k = \frac{\partial}{\partial \rho^k} [\tilde{f}(\rho)(\rho^k - \rho_0^k)\, \mathrm{e}^{\mathrm{i}\rho\cdot x}]|_{\rho=0}$$

is in the centre of the Lie algebra.

Therefore we summarize the results of this section.

Theorem 2. A $[\ ,\ ]_A$-Lie algebra, for which A is an entire analytic function and which has a trivial centre, is a generalized Moyal Lie algebra (1.7) and (1.9) and is isomorphic to the Moyal Lie algebra, via (1.10).

Theorems 1 and 2 establish the uniqueness of the Moyal and Poisson structures for functions defined on $\mathbb{R}^{2n}$, in a more general setting than previous works.

Acknowledgment

We would like to thank one of the referees for his valuable comments.

Appendix A

Here we show that (2.2) essentially implies (1.8). The following relations are easily obtained from (1.15) (cf (1.11))

$$B(0,\sigma) = B(\sigma,0) = B(0,0). \tag{A1}$$

$$b_s(\sigma,\sigma') - b_s(\sigma',\sigma'') + b_s(\sigma+\sigma',\sigma'') - b_s(\sigma,\sigma'+\sigma'') = b_a(\sigma',\sigma'') - b_a(\sigma,\sigma') + b_a(\sigma,\sigma'+\sigma'') - b_a(\sigma+\sigma',\sigma'')$$

from which cyclic permutation and addition give

$$b_a(\sigma+\sigma',\sigma'') + b_a(\sigma'+\sigma'',\sigma) + b_a(\sigma''+\sigma,\sigma') = 0 \tag{A2}$$

whereas the antisymmetric and symmetric parts in σ, σ'' give

$$b_a(\sigma,\sigma') - b_a(\sigma',\sigma'') + b_a(\sigma+\sigma',\sigma'') - b_a(\sigma,\sigma'+\sigma'') = 0 \tag{A3}$$

$$b_s(\sigma,\sigma') - b_s(\sigma',\sigma'') + b_s(\sigma+\sigma',\sigma'') - b_s(\sigma,\sigma'+\sigma'') = 0. \tag{A4}$$

Equations (A2) and (A3) imply the bilinearity of b_a, hence b_a is an exterior 2-form on the even-dimensional manifold Γ. Therefore, *if it is non-degenerate* then there is a canonical basis such that

$$b_a(\sigma,\sigma') = \mu\sigma' \wedge \sigma \qquad \mu \in \mathbb{C}. \tag{A5}$$

Acting on (A4) with $\partial_\sigma^2\partial_{\sigma'} - \partial_{\sigma'}^2\partial_\sigma$ we easily obtain that $(\partial_\sigma^2\partial_{\sigma'} - \partial_{\sigma'}^2\partial_\sigma)b_s(\sigma,\sigma')$ is a function of σ only, so that by the symmetry of b_s and (A1), we get $(\partial_\sigma - \partial_{\sigma'})\partial_{\sigma'}\partial_\sigma b_s(\sigma,\sigma') = 0$, which is readily integrated. Finally, by using (A1) in (A4) with $\sigma' + \sigma'' = 0$ and the symmetry of b_s, the solution takes the form

$$b_s(\sigma,\sigma') = -\chi(\sigma+\sigma') + \chi(\sigma) + \chi(\sigma'). \tag{A6}$$

This together with (A5) and (1.11) show that $B(\sigma,\sigma')$ is of the form (1.8), which was to be proved. When b_a *is degenerate, (A5) holds on* $\Gamma/\mathrm{Ker}\, b_a$.

Appendix B

Here we prove the following lemma used in section 3 with the notation introduced there.

Lemma. If $A(\sigma,\sigma') = h(\omega(\sigma,\sigma'))$ where ω is a 2-form, is an entire analytic function and it satisfies (1.16) and (1.17), then if A is not identically zero, $h(x)$ is either $c\sinh\mu x$ or cx, μ, c being constants.

Proof. Putting $\omega(\sigma,\sigma') = x$, $\omega(\sigma',\sigma'') = y$, $\omega(\sigma'',\sigma) = z$, (1.16) becomes

$$h(x-z)h(y) + h(y-x)h(z) + h(z-y)h(x) = 0.$$

Differentiating with respect to z at $z = 0$, we find

$$h'(0)h(y-x) = h(y)h'(x) - h'(y)h(x).$$

The case $h'(0) = 0$ is readily excluded, since in this case, the previous equation implies that $h(x) = 0$, given that h is odd in x. Putting $\tilde{h}(x) := h(x)/h'(0)$ we get

$$\tilde{h}(x+y) = \tilde{h}(y)\tilde{h}'(x) + \tilde{h}'(y)\tilde{h}(x).$$

Differentiating this successively with respect to x and y and equating the results, we get

$$\tilde{h}(y)\tilde{h}''(x) = \tilde{h}''(y)\tilde{h}(x).$$

If $h''(x)$ is not identically zero in any open region, then

$$\frac{\tilde{h}''(x)}{\tilde{h}(x)} = \frac{\tilde{h}''(y)}{\tilde{h}(y)} =: \mu = \text{constant}$$

everywhere† and the result follows. If $\tilde{h}''(x) = 0$ in an open region, then $h(x) = cx$ and by the analyticity of A this holds everywhere (cf the proof of theorem 2.11 in [19]). □

References

[1] Agarwal G S and Wolf E 1970 *Phys. Rev.* D **2** 2161
Agarwal G S and Wolf E 1970 *Phys. Rev.* D **2** 2187
Agarwal G S and Wolf E 1970 *Phys. Rev.* D **2** 2206
[2] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover) ch 4, section 14
[3] Wigner E P 1932 *Phys. Rev.* **40** 749
[4] Moyal J E 1949 *Proc. Cambridge Phil. Soc.* **45** 99
[5] Lee H-W 1995 *Phys. Rep.* **259** 147
[6] Hillery M, O'Conell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
Balazs N L and Jennings B K 1984 *Phys. Rep.* **104** 347
Takahashi K 1989 *Prog. Theor. Phys. Suppl.* **98** 109
[7] Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)
Gardiner C W 1985 *Handbook of Stochastic Methods* (Berlin: Springer)
[8] Balescu R 1963 *Statistical Mechanics of Charged Particles* (New York: Wiley)
Mori H, Oppenheim I and Ross J 1962 *Studies in Statistical Mechanics* vol I, ed J DeBoer and G E Uhlenbeck (Amsterdam: North-Holland)
Balescu R 1975 *Equilibrium and Non-equilibrium Statistical Mechanics* (New York: Wiley)
Kim Y S and Zachary W W 1987 *The Physics of Phase Space (Lecture Notes in Physics 278)* (Berlin: Spinger)
[9] Carruthers P and Zachariasen F G 1983 *Rev. Mod. Phys.* **55** 245
[10] Bakas I and Kakas A C 1987 *Class. Quantum Grav.* **4** L71
[11] Dunne G V 1988 *J. Phys. A: Math. Gen.* **21** 2321
[12] Krüger J G and Poffyn A 1976 *Physica* **85A** 84
[13] O' Connell R F and Wigner E P 1981 *Phys. Lett.* **85A** 121
Jagannathan R, Simon R, Sudarshan E C G and Vasuderan R 1987 *Phys. Lett.* A **120** 161
Davidović D M and Lalović D 1992 *Physica* **182A** 643
[14] Vey J 1975 *Comm. Math. Helv.* **50** 421
[15] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys. (NY)* **110** 61
[16] Lichnerowicz A 1980 *Lecture Notes in Mathematics 775* (New York: Springer) p 105
[17] DeWilde M and Lecompte P B A 1983 *Lett. Math. Phys.* **7** 487
[18] Flato M, Lichnerowicz A and Sternheimer D 1976 *CR Acad. Sci. Paris* A **283** 19
[19] Arveson W 1983 *Comm. Math. Phys.* **89** 77
[20] Fletcher P 1990 *Phys. Lett.* **248B** 323
[21] Gracia-Bondía J M and Várilly J C 1988 *J. Phys. A: Math. Gen.* **21** L879
Várilly J C and Gracia-Bondía J M 1989 *Ann. Phys. (NY)* **190** 107
[22] Galetti D and de Toledo Piza A F R 1988 *Physica* **149A** 267
[23] Alcalde C 1990 *J. Math. Phys* **31** 2672
Figueroa H, Gracia-Bondía J M and Várilly J C 1990 *J. Math. Phys.* **31** 2664
Gracia-Bondía J M and Várilly J C 1995 *J. Math. Phys.* **36** 2691

† Notice that in this case, by the analyticity of h'' its zeros are accumulations points of points where it does not vanish, and the above equation follows by continuity.

[24] Bidegain F and Pinczon G 1996 *Comm. Math. Phys.* **179** 295
Bidegain F and Pinczon G 1995 *Lett. Math. Phys.* **33** 231
Bakas I and Kakas A 1987 *J. Phys. A: Math. Gen.* **20** 3713
Landsman N P 1993 *Rev. Mod. Phys.* **5** 775

[25] Jacobson N 1984 *Basic Algebra* vol II (Delhi: Hindustan Publishing)

[26] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)

[27] Fulton W and Harris J 1991 *Representation Theory. A First Course* (New York: Springer)